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Supplementary Document: Curved Three-Director Cosserat Shells with Strong Coupling

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1. Full Cosserat model for curved shells

For quick reference, we reproduce the Cosserat model for curved shells with terms up to order $O(h^5)$ as presented by Nebel et al. [NSBN23], originally derived [BGMN19] that we implemented for comparison with the simplified plate model for flat reference configurations [Nef04; SNB16]. Overall, the energy density of the shell can be decomposed as

$$\Psi(\overline{\mathbf{E}},\boldsymbol{\Gamma}) = \Psi_{\text{memb}}(\overline{\mathbf{E}},\boldsymbol{\Gamma}) + \Psi_{\text{bend}}(\boldsymbol{\Gamma}), \tag{1}$$

with the strain measures as defined in Section 3.1 of the paper. The full energy depends on the mean curvature H, the Gaussian curvature K, the "second fundamental tensor" $\mathbf{b}$ and the "surface alternating pseudo-tensor" $\mathbf{c}$ which are defined in Section 2. The individual terms of the energy density up to order $O(h^5)$ are then given by

$$\begin{aligned}\Psi_{\text{memb}} = &\left(h - K\frac{h^3}{12}\right)\Psi_{\text{m}}(\overline{\mathbf{E}}) + \left(\frac{h^3}{12} - K\frac{h^5}{80}\right)\Psi_{\text{m}}(\overline{\mathbf{E}}\mathbf{b} + \mathbf{c}\boldsymbol{\Gamma}) \\ &+ \frac{h^3}{6}\Psi_{\text{mixt}}(\overline{\mathbf{E}}, \mathbf{c}\boldsymbol{\Gamma}\mathbf{b} - 2H\mathbf{c}\boldsymbol{\Gamma}) + \frac{h^5}{80}\Psi_{\text{mp}}(\overline{\mathbf{E}}\mathbf{b}^2 + \mathbf{c}\boldsymbol{\Gamma}\mathbf{b}),\end{aligned} \tag{2}$$

$$\begin{aligned}\Psi_{\text{bend}} = &\left(h - K\frac{h^3}{12}\right)\Psi_{\text{curv}}(\boldsymbol{\Gamma}) + \left(\frac{h^3}{12} - K\frac{h^5}{80}\right)\Psi_{\text{curv}}(\boldsymbol{\Gamma}\mathbf{b}) \\ &+ \frac{h^5}{80}\Psi_{\text{curv}}(\boldsymbol{\Gamma}\mathbf{b}^2),\end{aligned} \tag{3}$$

with

$$\begin{aligned}\Psi_{\text{mixt}}(\mathbf{A},\mathbf{B}) = &\mu(\operatorname{sym}\mathbf{A} : \operatorname{sym}\mathbf{B}) + \mu_c(\operatorname{skew}\mathbf{A} : \operatorname{skew}\mathbf{B}) \\ &+ \frac{\lambda\mu}{\lambda+2\mu}\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{B}\end{aligned} \tag{4}$$

$$\Psi_{\text{m}}(\mathbf{A}) = \mu\|\operatorname{sym}\mathbf{A}\|^2 + \mu_c\|\operatorname{skew}\mathbf{A}\|^2 + \frac{\lambda\mu}{\lambda+2\mu}(\operatorname{tr}\mathbf{A})^2 \tag{5}$$

$$\Psi_{\text{mp}}(\mathbf{A}) = \mu\|\operatorname{sym}\mathbf{A}\|^2 + \mu_c\|\operatorname{skew}\mathbf{A}\|^2 + \frac{\lambda}{2}(\operatorname{tr}\mathbf{A})^2, \tag{6}$$

and finally

$$\Psi_{\text{curv}}(\mathbf{A}) = \mu L_c^2\left(b_1\|\operatorname{sym}\mathbf{A}\|^2 + b_2\|\operatorname{skew}\mathbf{A}\|^2 + b_3(\operatorname{tr}\mathbf{A})^2\right). \tag{7}$$

By assuming a planar reference configuration, all terms scaled by the mean curvature H, the Gaussian curvature K and the second fundamental tensor $\mathbf{b}$ vanish and the plate energy described in Section 3.2 of the paper is recovered.

2. Surface curvature

The Cosserat shell model for curved initial configurations shown in Section 1 depends on the curvature of shell in its rest state. Here, we briefly present how to compute these values in the notation we used to describe the kinematic relations in Section 3.1 of the paper.

In classic shell models and in computer graphics, the surface curvature is often described using the shape operator or Weingarten map which can be represented by a matrix $\mathbf{S} \in \mathbb{R}^{2\times2}$ which is given by

$$\mathbf{S} = \mathbf{I}^{-1}\mathbf{II} = -(\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T\nabla_\omega\boldsymbol{n} = -\mathbf{J}^\dagger\nabla_\omega\boldsymbol{n}, \tag{8}$$

where $\boldsymbol{n}$ is the surface normal and $\mathbf{II} \in \mathbb{R}^{2\times2}$ represents the second fundamental form of the surface which can be computed as

$$\mathbf{II} = -\mathbf{J}^T\nabla_\omega\boldsymbol{n} \quad \text{or} \quad [\mathbf{II}]_{\alpha\beta} = \frac{\partial^2\boldsymbol{m}_0}{\partial\xi_\alpha\partial\xi_\beta}\cdot\boldsymbol{n}, \tag{9}$$

see [Pet23]. The Cosserat shell model instead uses the "second fundamental tensor" $\mathbf{b} \in \mathbb{R}^{3\times3}$ which encodes the same information as the shape operator and corresponds to the negative surface gradient of the normal field:

$$\mathbf{b} = -\nabla_{\mathcal{S}}\boldsymbol{n} = -\nabla_\omega\boldsymbol{n}\,\mathbf{J}^\dagger. \tag{10}$$

Evidently, $\mathbf{b}$ is closely related to the second fundamental form $\mathbf{II}$, in particular we have

$$\mathbf{b} = (\boldsymbol{a}^\alpha \otimes \boldsymbol{e}_\alpha)\mathbf{II}(\boldsymbol{e}_\beta \otimes \boldsymbol{a}^\beta) = (\mathbf{J}^\dagger)^T\mathbf{II}\,\mathbf{J}^\dagger. \tag{11}$$

With these definitions, we can compute the mean curvature H and the Gaussian curvature K as

$$H = \frac{1}{2}\operatorname{tr}\mathbf{S} = \frac{1}{2}\operatorname{tr}\mathbf{b} \quad \text{and} \quad K = \det\mathbf{S} = \frac{1}{2}(\operatorname{tr}(\mathbf{b})^2 - \operatorname{tr}(\mathbf{b}^2)). \tag{12}$$

In the constitutive model, the quantities H, K and $\mathbf{b}$ are only evaluated with respect to the undeformed reference configuration using the corresponding normal field at rest $\boldsymbol{n}_0$.

However, a straightforward finite element discretization of the above expressions is only accurate if the geometry is closely approximated by the discretization, e.g. if it can be represented well

by curved elements. If the reference configuration is piecewise affine or if linear triangle elements are used for the discretization (i.e. individual elements are planar) the expressions would evaluate to zero inside of the elements. In this case, it could be beneficial to instead apply results from discrete differential geometry and for example use expressions for the curvatures as presented by Wen and Barbič [WB23] depending on the mid-edge normals between elements or investigate a combined approach aware of element curvature and normal jumps between elements inspired by Le et al. [LDB*23].

Finally, the constitutive model also uses the "surface alternating pseudo-tensor" $\mathbf{c} \in \mathbb{R}^{3\times3}$ given by

$$\mathbf{c} = \frac{1}{\sqrt{\det \mathbf{I}}} (\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 - \boldsymbol{a}_2 \otimes \boldsymbol{a}_1) . \tag{13}$$

All three tensors $\mathbf{a} = \mathbf{J}\mathbf{J}^\dagger$, $\mathbf{b}$ and $\mathbf{c}$ are in the tangent plane of the surface as they do not contain any component normal to the surface [BGMN19].

3. Element basis functions and quadrature

The basis functions for a second-order Lagrange triangle element with vertices

$$\boldsymbol{p}_1 = [0,0]^T, \quad \boldsymbol{p}_2 = [0,1]^T, \quad \boldsymbol{p}_3 = [1,0]^T$$
$$\boldsymbol{p}_4 = [\frac{1}{2},0]^T, \quad \boldsymbol{p}_5 = [\frac{1}{2},\frac{1}{2}]^T, \quad \boldsymbol{p}_6 = [0,\frac{1}{2}]^T$$

are given by

$$\begin{aligned}
N_1(x,y) &= 2x^2 + 4xy - 3x + 2y^2 - 3y + 1 , \\
N_2(x,y) &= x(2x-1) , \\
N_3(x,y) &= y(2y-1) , \\
N_4(x,y) &= 4x(1-x-y) , \\
N_5(x,y) &= 4xy , \\
N_6(x,y) &= 4y(1-x-y) .
\end{aligned}$$

See, for example, the book by Wriggers [Wri08, Eq. 4.27].

A three-point Guassian quadrature rule of order two with all points in the interior of the unit triangle with vertices $[0,0]^T$, $[1,0]^T$ and $[0,1]^T$ is given by the quadrature points

$$\boldsymbol{p}_1 = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \end{bmatrix}, \boldsymbol{p}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \end{bmatrix},$$

with the weights

$$w_1 = \frac{1}{6}, w_2 = \frac{1}{6}, w_2 = \frac{1}{6} .$$

The weights are not normalized and already account for the area of the given unit triangle (i.e. multiplication by 0.5). See [Wri08, Table 4.3] or [Dun85].

4. Timings

For full timing information of the presented experiments see Table 1 on the following page.

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		Cells		# DOF		averaged		Runtime		Rel. cost of	
Scene	Variant	Type	#	disp.	rot.	dt	iter.	total	avg. step	assembly	LU
Twisting strip	-	Tri6	320	2187	615	10 ms	4.2	31.37 s	78 ms	12.7%	81.2%
Plate under load	10x1	Tri3	20	66	66	10 ms	1.5	0.86 s	4 ms	59.7%	33.5%
	20x2	Tri3	80	189	189	10 ms	1.5	2.17 s	11 ms	54.0%	35.1%
	40x4	Tri3	320	615	615	10 ms	1.9	5.50 s	28 ms	42.4%	49.8%
	80x8	Tri3	1280	2187	2187	10 ms	2.0	13.61 s	68 ms	33.6%	58.6%
	160x16	Tri3	5120	8211	8211	10 ms	2.0	40.30 s	201 ms	23.4%	72.8%
	320x32	Tri3	20480	31779	31779	10 ms	2.0	138.75 s	694 ms	21.7%	76.3%
	640x64	Tri3	81920	124995	124995	10 ms	2.0	477.72 s	2389 ms	19.2%	78.9%
	10x1	Tri6	20	189	66	10 ms	2.4	2.00 s	10 ms	58.6%	32.6%
	20x2	Tri6	80	615	189	10 ms	2.3	6.42 s	32 ms	28.6%	51.9%
	40x4	Tri6	320	2187	615	10 ms	2.0	9.38 s	47 ms	21.9%	64.1%
	80x8	Tri6	1280	8211	2187	10 ms	2.0	30.96 s	155 ms	13.9%	78.2%
	160x16	Tri6	5120	31779	8211	10 ms	2.0	110.05 s	550 ms	13.1%	83.5%
	320x32	Tri6	20480	124995	31779	10 ms	2.0	371.19 s	1856 ms	10.9%	87.4%
	640x64	Tri6	81920	495747	124995	10 ms	2.0	1380.15 s	6901 ms	8.8%	89.7%
Cylinder coiling	4x1	Tri3	8	30	30	10 ms	1.3	2.65 s	2 ms	47.4%	36.9%
	8x2	Tri3	32	81	81	10 ms	1.4	7.35 s	6 ms	65.9%	25.4%
	16x4	Tri3	128	255	255	10 ms	1.6	12.58 s	10 ms	43.4%	38.5%
	32x8	Tri3	512	891	891	10 ms	1.6	35.44 s	30 ms	36.8%	49.8%
	64x16	Tri3	2048	3315	3315	10 ms	1.5	99.03 s	82 ms	28.2%	64.0%
	128x32	Tri3	8192	12771	12771	10 ms	1.5	293.21 s	244 ms	21.7%	73.6%
	256x64	Tri3	32768	50115	50115	10 ms	1.5	996.39 s	830 ms	21.5%	76.1%
	4x1	Tri6	8	81	30	10 ms	1.6	5.82 s	5 ms	48.0%	27.9%
	8x2	Tri6	32	255	81	10 ms	1.6	14.07 s	12 ms	51.6%	31.4%
	16x4	Tri6	128	891	255	10 ms	1.6	29.49 s	25 ms	30.7%	49.0%
	32x8	Tri6	512	3315	891	10 ms	1.6	69.91 s	58 ms	21.7%	68.9%
	64x16	Tri6	2048	12771	3315	10 ms	1.5	218.10 s	182 ms	14.4%	79.7%
	128x32	Tri6	8192	50115	12771	10 ms	1.5	750.85 s	625 ms	12.8%	85.1%
	256x64	Tri6	32768	198531	50115	10 ms	1.5	2657.76 s	2213 ms	10.5%	87.5%
Lotus flower	4x4	Tri6	32	243	75	10 ms	1.7	16.3 s	16 ms	39.5%	25.0%
	8x8	Tri6	128	867	243	10 ms	1.6	29.28 s	29 ms	27.8%	43.3%
	16x16	Tri6	512	3267	867	10 ms	1.6	71.57 s	71 ms	17.3%	60.8%
	32x32	Tri6	2048	12675	3267	10 ms	1.6	212.07 s	212 ms	13.0%	77.0%
	64x64	Tri6	8192	49923	12675	10 ms	1.6	651.62 s	651 ms	9.2%	88.8%
	100x100x1 (CG solver)	Tet4	60000	61206	61206	10 ms	1.3	523.65 s	524 ms	43.0%	51.9%
Bunny	$h = 10$ mm, KL	Tri3	6612	9924	-	10 ms	1.3	74.69 s	149 ms	19.0%	75.9%
	$h = 3.75$ mm, KL	Tri3	6612	9924	-	1 ms	1.9	1213.09 s	243 ms	23.3%	69.3%
	$h = 1.5$ mm, KL	Tri3	6612	9924	-	1 ms	3.0	1896.03 s	379 ms	22.3%	71.1%
	$h = 10$ mm	Tri6	6612	39678	9924	10 ms	1.3	254.27 s	508 ms	10.2%	82.2%
	$h = 3.75$ mm	Tri6	6612	39678	9924	1 ms	1.7	3277.94 s	656 ms	9.1%	85.9%
	$h = 1.5$ mm	Tri6	6612	39678	9924	1 ms	3.0	5579.10 s	1116 ms	9.1%	85.9%
Curvature modes	Top Left	Tri6	64	495	153	10 ms	1.6	9.82 s	20 ms	12.5%	39.3%
Armadillos	Left	Tri6	15482	92898	23229	5 ms	1.7	1113.99 s	1391 ms	7.7%	89.4%
Buckling circle	Inner	Tri6	120	783	213	10 ms	1.7	17.47 s	22 ms	17.7%	60.7%
	Outer	Tri6	120	783	213	10 ms	2.0	19.65 s	25 ms	16.8%	62.3%
Möbius strip	Plate model	Tri6	80	600	180	1 ms	1.8	71.53 s	18 ms	28.7%	68.3%
	Shell model	Tri6	80	600	180	1 ms	1.8	76.35 s	19 ms	28.5%	61.5%
Dumpling	Plate model	Tri6	1536	9411	2403	10 ms	2.1	126.46 s	210 ms	7.1%	75.2%
	Shell model	Tri6	1536	9411	2403	10 ms	2.1	140.78 s	234 ms	16.2%	67.6%
Papyrus scroll	-	Tri6	400	2583	693	9.9 ms	4.5	272.08 s	180 ms	21.5%	56.1%

Table 1: *Timing information on all experiments were obtained on a workstation system with an AMD Ryzen Threadripper PRO 5975WX CPU (32 cores,* 3.6 GHz*) and* 256 GB *of DDR4 8-channel RAM. For the experiments "Lotus flower", "Curvature modes" and "Armadillos" the quantities are given only for a single mesh (i.e. one plate/one armadillo) as the timings are approximately identical between all mesh instances used for the scenes. The element types "Tri3", "Tri6" and "Tet4" denote first- and second-order Lagrangian triangle elements and first-order Lagrangian tetrahedrons, respectively. Rotational degrees of freedom are always discretized using first-order elements. The "#DOF" columns "disp." and "rot." show the total number of (scalar) displacement and rotational degrees of freedom per simulation, respectively. The column "iter." shows the average number of Newton iterations required for convergence per time step. The last columns show respectively the cost of the matrix assembly (including the evaluation of the material model and its derivatives) and the LU decomposition linear solves relative to the total runtime of the simulation. The remaining, unaccounted percentage includes (depending on the experiment): collision detection, interpolation to the linear rendering mesh and file output. Note that all shell simulations used LU decomposition to solve linear systems while the volumetric lotus flower simulation used a CG solver as described by [LFJ*23].*